PHYSICAL REVIEW E **71**, 060301(R) (2005)

Jamming transition in two-dimensional hoppers and silos

Kiwing To
Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan 115, Republic of China
(Received 4 February 2005; published 17 June 2005)

Jamming of monodisperse metal disks flowing through two-dimensional hoppers and silos is studied experimentally. Repeating the flow experiment M times in a hopper or silo (HS) of exit size d, we measure the histograms $h(n)$ of the number of disks n through the HS before jamming. By treating the states of the HS as a Markov chain, we find that the jamming probability $J(d)$, which is defined as the probability that jamming occurs in a HS containing m disks, is related to the distribution function $F(n)\equiv(1/M)\Sigma_{s=n}^{s=\infty}h(s)$ by $J(d)=1-F(m)=1-e^{-\alpha(m-n_o)}$. The decay rate α, as a function of d, is found to be the same for both hoppers and silos with different widths. The average number of disks $N\equiv1/\alpha=\langle n\rangle$ passing through the HS can be fitted to $N=Ae^{Bd^2}$, $N=Ae^{B/(d_c-d)}$, or $N=A(d_c-d)^{-\gamma}$. The implications of these three forms for N to the stability of dense flow are discussed.

DOI: 10.1103/PhysRevE.71.060301 PACS number(s): 45.70.Qj, 45.70.Mg, 45.70.Vn

Many materials (e.g., sand and coal ore) exist in granular form in which the grains can be considered as hard objects that exert a repulsive force on each other only when they are in contact. The handling of these granular materials is of great importance in the chemical, agricultural, food, and pharmaceutical industries [1]. Although granular materials may flow like ordinary fluids, our basic understanding of granular flow is much poorer than that of the latter [2]. Despite the fact that the motion of each grain follows simple deterministic physical laws, the spatial and temporal averaging processes that lead to the hydrodynamic equations for ordinary fluids are inapplicable to granular flow in most cases due to the dissipative nature of the interparticle interaction. Without the equivalence of the Navier-Stokes equation, our understanding of granular flow is mainly phenomenological and relies heavily on experimental observations. In the simple situation of gravity driven granular flow in two-dimensional pipes and channels, it is known that the flow may be in the dilute, dense, or jammed state. Transitions among these flow states can be induced by changing the physical dimensions, the geometry, or the roughness of the confining boundaries [3–5]. However, the physical properties of these flow states and the nature of these transitions are not clearly known.

In our previous study, we examined the particle configurations in the jammed states of a two-dimensional hopper and established that the dense-to-jam transition was due to the formation of a permanent arch that blocked the flow at the hopper exit. A theoretical model based on the statistics of the arch was proposed to calculate the jamming probability $J(d)$ as a function of the hopper exit size d [6,7]. Intuitively, as long as $J(d)$ is finite, jamming is bound to occur if there is an infinite supply of particles. In fact, it has been reported [8] that $J(d)$ does increase with the number of disks in the hopper. So it is possible that the dense flow may not be a real steady state. In addition, while the system size does play a role in the dilute-to-dense transition, it has been suggested that the dense-to-jam transition depends on the particle size only [3].

In this paper, we report an experimental study of jamming in two-dimensional hoppers and silos. We find that $J(d)$ is related to the statistics of the number of particles passing through the hopper before jamming. By treating the states of the hopper and silo (HS) as a Markov chain, an expression in terms of the total number of disks m in the HS for $J(d)$ is derived. Surprisingly, our data show that jamming in a two-dimensional silo is the same as that in a two-dimensional hopper. Furthermore, the jamming statistics are independent of the silo width, proving that jamming is indeed a local transition process that depends only on the ratio between the exit size and the grain size.

When studying the dilute-to-dense transition, both the dilute and dense states are steady states with particles moving in and out with constant flux. Hence, statistical properties of these two states can be obtained from temporal measurements. In contrast, once the flow is jammed, the system is locked in some particular quiescent state with no further particle motion. Hence, an ensemble of the jammed state can only be obtained by repeated experiments under identical conditions.

Our experiments are performed using a two-dimensional container in which the two-dimensional hopper is constructed. Figure 1 is a schematic diagram of the container and the hopper. The container is made of two transparent acrylic boards with a 3.5 mm gap between them. One thousand copper disks of 5 mm diameter and 3 mm thick are put into the container. The disks are coated with nickel to reduce friction and the repose angle of the disks in the two-dimensional container is found to be 30°. Two aluminum plates (W_1,W_2) at the top portion (T) of the container form a 10-cm-wide channel where the disks stay before falling down. A pair of brass plates (V_1,V_2), a pair of transparent triangular acrylic plates (A_1,A_2), and another pair of triangular brass plates (H_1,H_2) are placed in the central portion of the container. The assembly V_1, A_1, and H_1 (V_2, A_2, and H_2) are glued together to form the left (right) wall of a two-dimensional hopper with the incline edge A_1 (A_2), making an angle $\phi=52°$ to the horizontal. The horizontal positions of the left and right walls of the hopper can be adjusted independently to set the hopper exit size d. By removing A_1 and

 ©2005 The American Physical Society

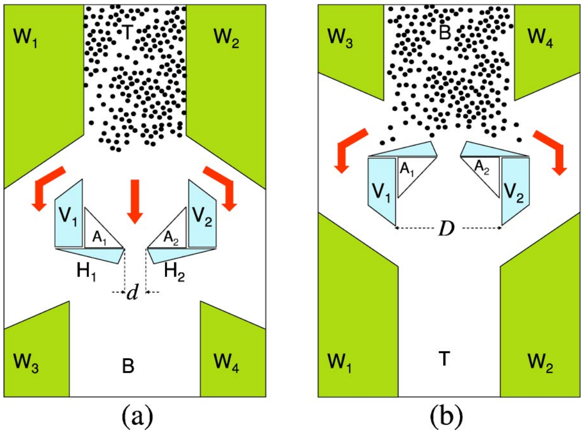

FIG. 1. Schematic diagram of the two-dimensional hopper in the two-dimensional container. (a) The experiment starts with the container in the "up" position when the disks flow toward the hopper $(V_1, H_1, A_1, V_2, H_2, A_2)$. (b) Disks flow back to the top (T) at the "down" position. The arrows show the paths of the disks.

A_2, we can turn the two-dimensional hopper into a two-dimensional silo of width D.

The container is securely mounted on a platform that flips the container to the "up" and "down" positions. Figure 1(a) is a schematic diagram of the container in the "up" position when the disks fall from the top toward the hopper under gravity. Disks can flow into the hopper and then fall to the space (B) between a pair of metal plates $(W_3$ and $W_4)$ at the bottom of the container. Since the hopper has a finite capacity, the hopper may overflow and some disks will fall on top of W_3 and W_4 instead of falling into B. After 10 s, the platform flips the container to the "down" position as shown in Fig. 1(b). At this position, all the disks return to T. Then the platform flips the container back to the "up" position and the experiment repeats.

To capture the jamming configuration at the hopper exit and to count the number of disks falling through, we place a light box (L) at the back of the container and two cameras (CCDA, CCDB) at the front for capturing images of the hopper as well as that of the bottom [see Fig. 2(a)]. A contact switch is installed to detect the moment when the hopper arrives at the up position. Then the video signals from the cameras are digitized by a frame grabber (Data Translation, model DT2851) of a computer (PC) 10 s after the container has arrived at the "up" position. Figure 2(b) shows a typical image taken by camera CCDA when the flow is jammed in a hopper of $D=20$ and $d=3.68$. (D and d are given in units of the disk diameter.) Figure 2(c) is the corresponding image captured by camera CCDB for the disks that fall through the hopper to the bottom of the container. From these two images we can determine, in each experiment, if jamming has occurred and the number of disks falling through the hopper.

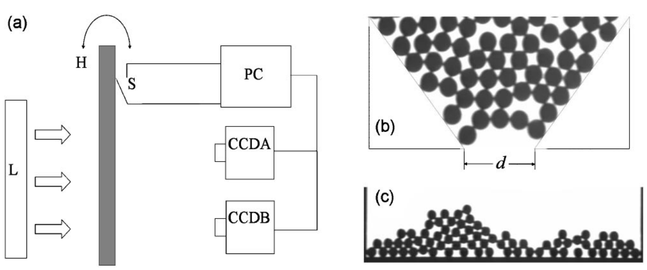

FIG. 2. (a) Schematic diagram of the setup for capturing disk configuration images. (b) Image of the jammed hopper. (c) Image of the fallen disks in the bottom of the container.

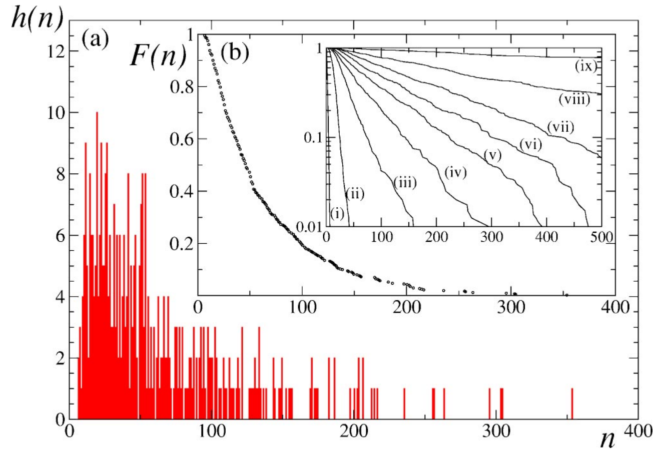

FIG. 3. (a) Histogram $h(n)$ of the number of disks n fallen before jamming occurs for the hopper of exit size $d=3.68$. (b) The distribution function $F(n)$ obtained from $h(n)$ at this hopper exit size. The inset of (b) is the semilog plots of $F(n)$ for (i) $d=1.48$, (ii) 3.27, (iii) 3.68, (iv) 3.80, (v) 4.16, (vi) 4.24, (vii) 4.63, (viii) 4.88, (ix) 5.17.

For each hopper of exit size d, we repeat the experiment $M=400$ times and calculate the jamming probability $J(d)$ as the fraction of jamming events. In addition we construct a histogram $h(n)$ for the number of disks n that fall through the hopper in each set of experiments. From $h(n)$ we obtain the distribution function $F(n)$, which is defined as the fraction of events such that at least n disks have fallen through, by the following expression:

$$F(n) = \frac{1}{M}\sum_{s=n}^{s=\infty} h(s). \tag{1}$$

Figure 3(a) shows the histogram $h(n)$ for the hopper of $d=3.68$ and Fig. 3(b) is the plot of the distribution function $F(n)$ calculated by Eq. (1). One can see that $F(n)$ decays exponentially with n, i.e.,

$$F(n) = e^{-\alpha(n-n_o)}. \tag{2}$$

Fitting the data to the above equation gives $n_o=5.03$ and the decay rate $\alpha=0.0169$. We repeat the experiments for d in the range from 1.48 to 5.92 and we find that $F(n)$ also decays exponentially [as shown in the inset of Fig. 3(b)] with n_o fluctuating between 0 and 50.

The exponential decay form for $F(n)$ in Eq. (2) can be understood in the following way. At the beginning of the experiment, disks drop from the top (T) to the hopper under gravity. Because of the acceleration the disk density decreases as the disks flow toward the hopper. When the first few disks reach the hopper exit, the system is in the dilute flow regime because the density is too low to trigger the dilute-to-dense transition [3]. Only when enough disks have arrived at the exit may a steady dense flow be established. Hence, the quantity n_o is the number of disks that fall through the hopper in this transient dilute flow period.

When the system is in the steady dense flow regime, we may treat the hopper as a device that is either flowing (i.e., at least one more disk can get through) or jammed (i.e., no more disks can get through). Assume that the outflow of disks through the exit may be considered as a succession of independent events. Let p be the transition probability that the hopper changes from the flowing state to the jammed state after the next disk has passed through. Then the transition probability that the hopper remains in the flowing state is $q=1-p$. Once the hopper transits from the flowing state to the jammed state, no more disks can get through because the exit has been blocked. So the transition probability for the hopper to change from the jammed state to the flowing state is zero and that to remain in the jammed state is one. Hence we have a Markov process with a transition probability matrix given by $T=\begin{pmatrix} q & p \\ 0 & 1 \end{pmatrix}$. The probability that the hopper, which has been flowing after n_o disks, remains flowing after another $n-n_o$ disks get passed is $T_{11}^{(n-n_o)}=q^{n-n_o}$. This is just the distribution function $F(n)$ measured in our experiments. Hence,

$$F(n) = q^{n-n_o} = e^{(n-n_o)\ln q}. \tag{3}$$

Comparing the above equation to Eq. (2), the decay rate is found to be $\alpha=-\ln q=-\ln(1-p)$.

To find the relation between the jamming probability $J(d)$ and the jamming transition probability p, we note that if there are only m disks in the container, only those events with less than m disks passing through the hoppers are considered jammed. Therefore $J(d)$ is just the complement of $F(m)$, i.e., $J(d)+F(m)=1$, or

$$J(d) = 1-(1-p)^{m-n_o} = 1-e^{-\alpha(m-n_o)}. \tag{4}$$

To proceed further, we extract α from $F(n)$. We find that the data decreases linearly in the semilog plot of α versus d^2 as shown in the inset of Fig. 4. Hence, we fit the data to the empirical form

$$\alpha = Ae^{-Bd^2} \tag{5}$$

with two fitting parameters A and B. (It will be shown later that our data can also be fitted to other functional forms with three fitting parameters.) Setting the simple exponential-square form of α to Eq. (4), we have

$$J(d) = 1-e^{-(m-n_o)Ae^{-Bd^2}}. \tag{6}$$

On the other hand, the jamming probability $J(d)$ can be obtained readily by counting the number of jamming events using the images captured by camera CCDA. Figure 5 shows the measured $J(d)$ and that calculated by expression (6) using $n_o=25$, $m=500$, $A=0.846$, and $B=0.275$. The good agreement shows that jamming can indeed be described by a Markovian process.

The fact that the simple exponential form of $F(n)$ can be explained by a Markovian process is somewhat surprising. While the outflow of disks through the exit may be considered as a series of independent events, the jamming process requires disks to be in some particular configurations (arches at the exit) to block the flow. Hence, around the jamming transition, correlations are expected to be important. Presumably, before jamming actually occurs, the length scale of these correlations remains smaller than the exit size so that the disks flowing through the exit can be considered as composed of independent groups of disks with the correlation length scale.

Hou *et al.* suggested that there was only one dominant length scale (the exit size d) in the dense-to-jam transition. To check this speculation, we investigate the jamming statistics of two-dimensional silos of different width D. This can

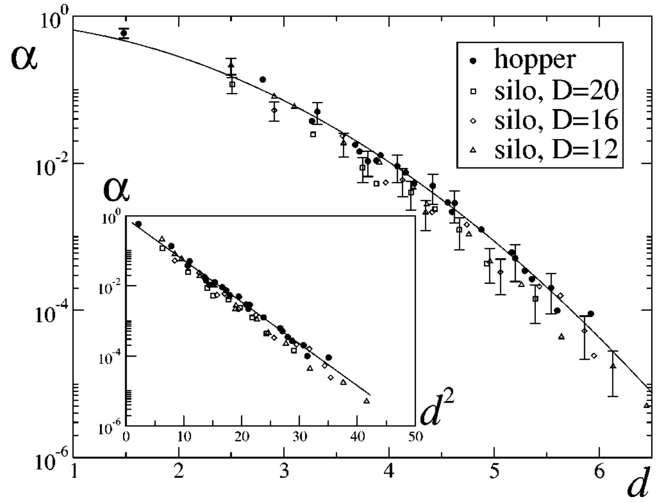

FIG. 4. Variation of the decay rate α with hopper or silo exit d. The solid line is the fitted curve $\alpha=Ae^{-Bd^2}$ with $A=0.846$ and $B=0.275$. The inset shows the same data and the fitted curve with d^2 plotted in the x axis.

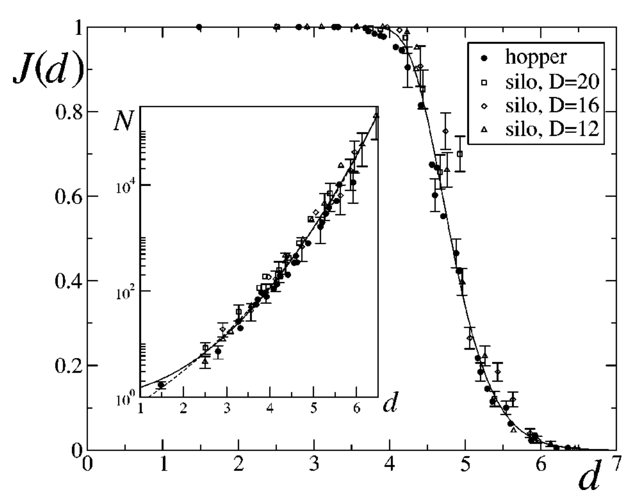

FIG. 5. Jamming probability $J(d)$ versus hopper or silo exit d. The solid line is the fitted curve using Eq. (6). The inset shows the mean number of disks N through the hopper or silo before jamming. The solid line in the inset is from the exponential-square form for N. The dashed curve is from the power law and the exponential-reciprocal functional forms which are indistinguishable from each other in this range of d.

be achieved in our container by adjusting the positions of the walls (V_1, V_2) after removing the transparent acrylic plates (A_1, A_2). The decay rate α and the jamming probability $J(d)$ measured at different d for silos of $D=20$, 16, and 12 are included in Figs. 4 and 5, respectively. One can see that both α and $J(d)$ are independent of D. This provides evidence that the jamming transition depends only on the exit size d but not on the silo width D. Moreover, α and $J(d)$ of the silos are the same as those of the hoppers with the same d within experimental uncertainties. Since a silo can be considered as a hopper of $\phi=0°$, such observation implies that the jamming transition is independent of the shape of the hopper. This is in agreement with Ref. [6], which reports that $J(d)$ is the same for hoppers of $\phi=34°$ and $60°$.

It should be pointed out that the decay rate α is the hopper or silo (HS) failure rate, and its reciprocal is the mean number of disks N through the HS before jamming. The inset of Fig. 5 shows how rapidly N increases with d. Nevertheless, the exponential-square form of $N=A^{-1}e^{Bd^2}$ implies that N remains finite for finite d. In other words, if there is unlimited supply of particles to flow, the hopper will jam eventually. For example, at $d=6$, $\alpha \approx 4\times10^{-5}$. Hence after $m=10^5$ particles have passed, the probability to find the hopper in the jammed state is $J(d)=1-\exp(-10^5\times4\times10^{-5})=0.98$ from Eq. (4). In the experiment performed by Hou *et al.*, at $d=6$, the dense flow rate was 4 g/s (see Fig. 2 in Ref. [3]) which was equivalent to 1000 spheres flowing out of their channel per second. Therefore, the dense flow would most probably be jammed if they could have waited for $10^5/10^3=100$ s. This means that the dense flow cannot be a real steady state at this exit.

The above implication is contrary to our intuition that a hopper should not jam when its exit size is much larger than the particle size. Indeed, recent experimental studies on jamming in a three-dimensional silo by Zuriguel [9,10] did reveal the existence of a critical exit d_c beyond which no jamming should occur. Although there is no evidence that a two-dimensional hopper or silo should behave similarly in this aspect, our data can also be fitted equally well to a power law: $N=A(d_c-d)^{-\gamma}$ with $A=10^{10}$, $\gamma=11.2$, and $d_c=9.09$; or to an exponential-reciprocal form: $N=Ae^{B/(d_c-d)}$ with $A=8\times10^{-8}$, $B=201$, and $d_c=13.5$. However, the fitted curves of these two forms and that for the exponential-square form are numerically indistinguishable within our experimental range $1<d<6.5$ (see the inset of Fig. 5). In order to tell if a critical exit size exists for two-dimensional hoppers or silos, more data at large exit sizes will be needed. Currently, we are building a setup in which the particles that fall out can be circulated back to the entrance of the hopper. Then we can perform our experiments close to the critical exit size d_c, if it does exist.

To summarize, we have studied experimentally the jamming transition in two-dimensional hoppers and silos. We find that the jamming probability $J(d)$ is related to the distribution function $F(n)$ for the number of particles n through the hopper or silo before the dense-to-jam transition occurs. While the jamming probability depends only on the exit size d, it is the same for hopper and silo, and regardless of width D. We also discuss the possibility of a critical exit size d_c beyond which the hopper or silo will never be jammed.

The author would like to thank Dr. C. K. Chan, Professor P. Y. Lai, and Dr. K-t Leung for very helpful discussion. This work is supported by the National Science Council of the Republic of China under Grant No. NSC-93-2112-M-001-015.

[1] R. M. Nedderman, *Statics and Kinematics of Granular Materials* (Cambridge University Press, London, 1992).
[2] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
[3] M. Hou, W. Chen, T. Zhang, K. Lu, and C. K. Chan, Phys. Rev. Lett. **91**, 204301 (2003).
[4] J. C. Tsai, G. A. V. W. Losert, and J. P. Gollub, Phys. Rev. E **65**, 11306 (2001).
[5] S. Horluck and P. Dimon, Phys. Rev. E **63**, 031301 (2001).
[6] K. To, P.-Y. Lai, and H. Pak, Phys. Rev. Lett. **86**, 71 (2001).
[7] K. To and P.-Y. Lai, Phys. Rev. E **66**, 011308 (2002).
[8] K. To, P.-Y. Lai, and H. Pak, Physica A **315**, 174 (2002).
[9] I. Zuriguel, L. A. Pugnaloni, A. Garcimartin, and D. Maza, Phys. Rev. E **65**, 030301(R) (2003).
[10] I. Zuriguel, A. Garcimartin, D. Maza, L. A. Pugnaloni, and J. M. Pastor, e-print cond-mat/0407761.